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HAMILTON DYNAMICS AND H -PLANAR CURVES

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Abstract

An important example of Hamilton flows on Kähler manifold are H -planar flows: all their trajectories are H -planar curves (complex analog of geodesics). The equation which has to obey the Hamiltonian of H -planar flow is received and the method of finding general solution of this equation is proposed.

Trajectories of charged particles in magnetic fields of special form on Kähler manifolds of constant holomorphic sectional curvature are studied. Using the fact that Kähler manifolds of constant holomorphic sectional curvature admit H -projective mapping on flat space \mathbf{C}^n the equation of particle motion is reduced to an ordinary differential equation of second order.

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1 Introduction

The purpose of this paper is to investigate Hamilton systems on Kähler manifolds from the point of view of differential geometry. Kähler manifolds have exceptional significance in geometry and physics since they have both Riemannian and symplectic structures and these two structures are interconnected. Because of this fact one can consider two dynamics on a Kähler manifold M : one dynamics is defined by the Riemannian structure and all its trajectories are geodesics (or H -planar curves [9, 11] which can be considered as a natural analog of geodesics in complex case) and another dynamics is connected with symplectic structure on M : it is defined by the Hamilton equations.

It is of interest to answer the following question: when this two dynamics coincides, namely, what are the conditions which Hamiltonian \mathcal{H} and Kähler metric g have to obey in order to trajectories of Hamilton flow be geodesics (or in more general case H -planar curves). The first part of this paper is devoted to the solution of this problem (see Theorem 2, Sec. 2).

At the second part a particular case of H -planar curves which has important physical applications are considered. These H -planar curves can be interpreted as trajectories of charged particles in a magnetic field of special form (Kähler magnetic fields). Particle motion in this case is ruled by the system of $2n$ differential equation of second order and solution of this system in general case is a complicated problem. But if considered Kähler manifold have constant holomorphic sectional curvature it can be shown (see Sec.4) that the number of equations can be reduced to one. It gives the possibility to investigate the structure of trajectories space by qualitative methods or receive the solution numerically.

We start from recalling some relevant facts of differential geometry of Kähler and symplectic manifolds [3, 8, 11].

A Hermitian manifold M with metric g and complex structure J is called *Kähler manifold* if its fundamental form ω (defined by the condition $\omega(X, Y) = g(JX, Y)$ for any two vector fields X, Y on M) is closed $d\omega = 0$. A Kähler manifold is said to be *properly Kähler* if its metric is positively definite (Riemannian). Because of integrability of complex structure J it is possible to introduce on Kähler manifold (M, g, J) , $\dim_{\mathbf{R}} M = 2n$ local complex coordinates z^α, \bar{z}^α , $\alpha = 1, \dots, n$ such that the components of metric and complex structure have the following form [8]

$$g_{\alpha\bar{\beta}} = \overline{g_{\bar{\alpha}\beta}} = \partial_\alpha \bar{\partial}_{\bar{\beta}} \Phi, \quad g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0, \quad (1)$$

$$J^\alpha_\beta = -J^\alpha_{\bar{\beta}} = i\delta^\alpha_\beta, \quad J^\alpha_{\bar{\beta}} = J^\alpha_\beta = 0 \quad (2)$$

where bar denotes complex conjugation and real-valued function Φ on M is *Kähler potential* of metric g .

Since fundamental 2-form ω of Kähler manifold M is non-degenerate (M, ω) is *symplectic manifold* [3]. For any smooth function f on M its *gradient vector*

field $\text{grad}(f)$ and *Hamiltonian vector field* $\text{ad}(f)$ are defined by the conditions $g(X, \text{grad}(f)) = df(X)$ and $\omega(X, \text{ad}(f))\omega = df$ where X is a vector field on M . It is easy to see that vector fields $\text{grad}(f)$ and $\text{ad}(f)$ are orthogonal at any point $p \in M$ and for any function f : $g(\text{ad}(f), \text{grad}(f)) = -\omega(\text{ad}(f), \text{grad}(f)) = 0$. The components of these vector fields in local coordinates x^i on M can be written in the form

$$\text{grad}(f)^i = g^{ik} \partial_k f, \quad \text{ad}(f)^i = \omega^{ki} \partial_k f \quad (3)$$

where g^{ik} and ω^{ik} are components of matrices inverse to the matrices (g_{kj}) , (ω_{kj}) of metric and fundamental 2-form. Dynamical system γ defined by *Hamilton equation*

$$\frac{d\gamma}{dt} = \text{ad}(\mathcal{H}) \quad (4)$$

on M is said to be *Hamilton flow* with *Hamiltonian* \mathcal{H} . Curves γ_t are called trajectories of dynamical system γ .

A curve $\gamma : [0, 1] \rightarrow M$ on a Kähler M is called *H-planar curve* if $\chi \equiv d\gamma/dt$ obeys the equation

$$\nabla_\chi \chi = a(t)\chi + b(t)J(\chi) \quad (5)$$

where $a(t)$, $b(t)$ are some smooth functions of t and ∇ is Riemannian connection of Kähler metric g . *H-planar curves* are natural analog of geodesics in complex case, namely, *H-planar curves* are real realizations of geodesics in the space over complex algebra [13]. We shall call Hamilton flow *H-planar flow* if all its trajectories are *H-planar curves*.

Let (M, g, J) and (M', g', J') be two Kähler manifolds. Diffeomorphism $f : M \rightarrow M'$ is called *H-projective mapping* [9] if an image $f \circ \gamma$ of any *H-planar curve* γ on M is *H-planar curve* on M' . In order to mapping $f : M \rightarrow M'$ be *H-projective* it is necessarily preserving of the complex structure J : $f_* \circ J = J' \circ f_*$ where f_* is differential of f .

For future considerations we shall need the following theorem

Theorem 1 [11] *Kähler manifold (M, g) is H-projectively flat, i.e. it admits H-projective mapping f on flat space $\mathbf{C}^n \cong \mathbf{R}^{2n}$, if and only if its holomorphic sectional curvature k is constant.*

2 H-planar flows on a Kähler manifolds

Let us consider Kähler manifold M with metric g , fundamental 2-form ω and let γ be Hamilton flow on M . Since *H-planar curves* are mostly natural analog of geodesics in complex case, *H-planar flows* form the important class of Hamilton flows on Kähler manifolds.

The problem arises: *what are the conditions which has to obey Hamiltonian function \mathcal{H} in order to correspondent Hamilton flow be H-planar?* The solution of this problem is given by the following

Theorem 2 *Hamilton flow γ with Hamiltonian \mathcal{H} on Kähler manifold M is H -planar if and only if \mathcal{H} obeys the condition*

$$\nabla_{\text{grad}(\mathcal{H})} \text{grad}(\mathcal{H}) = B \text{grad}(\mathcal{H}) - A J \text{grad}(\mathcal{H}) \quad (6)$$

where A and B are real-valued functions smooth on M . Each trajectory γ_t of the flow γ is H -planar curve on M satisfying the equation (5) where $a(t) = A(\gamma_t)$ and $b(t) = B(\gamma_t)$.

Proof. If a Hamilton flow γ is H -planar then any its trajectory γ_t is H -planar curve and obeys the equation (5). At the same time γ satisfies the Hamilton equation (4). In a local complex coordinates z^α, \bar{z}^α , $\alpha = 1, \dots, n$ on M these two equations take the form

$$\frac{d^2 z^\alpha}{dt^2} + \Gamma_{\mu\nu}^\alpha \frac{dz^\mu}{dt} \frac{dz^\nu}{dt} = a(t) \frac{dz^\alpha}{dt} + b(t) J_\mu^\alpha \frac{dz^\mu}{dt}, \quad (7)$$

$$\frac{dz^\alpha}{dt} = \omega^{\bar{\mu}\alpha} \partial_{\bar{\mu}} \mathcal{H} \quad (8)$$

where $\Gamma_{\mu\nu}^\alpha = \overline{\Gamma_{\bar{\mu}\bar{\nu}}^{\bar{\alpha}}}$, $\Gamma_{\bar{\mu}\bar{\nu}}^\alpha = \Gamma_{\mu\nu}^\alpha = \Gamma_{\bar{\mu}\bar{\nu}}^{\bar{\alpha}} = \Gamma_{\mu\nu}^{\bar{\alpha}} = 0$ are Cristoffel symbols of metric g .

Differentiating (8) we find

$$\begin{aligned} \frac{d^2 z^\alpha}{dt^2} &= \partial_\nu \omega^{\bar{\mu}\alpha} \omega^{\nu\bar{\rho}} \partial_{\bar{\mu}} \mathcal{H} \partial_{\bar{\rho}} \mathcal{H} + \partial_{\bar{\nu}} \omega^{\bar{\mu}\alpha} \omega^{\bar{\nu}\rho} \partial_{\bar{\mu}} \mathcal{H} \partial_\rho \mathcal{H} + \\ &\quad \omega^{\bar{\mu}\alpha} \omega^{\nu\bar{\rho}} \partial_{\bar{\rho}} \mathcal{H} \partial_{\bar{\mu}\nu} \mathcal{H} + \omega^{\bar{\mu}\alpha} \omega^{\bar{\nu}\rho} \partial_\rho \mathcal{H} \partial_{\bar{\mu}\bar{\nu}} \mathcal{H}. \end{aligned} \quad (9)$$

From (7), (8) and (9) it follows

$$\begin{aligned} &\Gamma_{\mu\nu}^\alpha \omega^{\mu\bar{\rho}} \omega^{\nu\bar{\tau}} \partial_{\bar{\rho}} \mathcal{H} \partial_{\bar{\tau}} \mathcal{H} - \partial_\nu \omega^{\bar{\mu}\alpha} \omega^{\nu\bar{\rho}} \partial_{\bar{\mu}} \mathcal{H} \partial_{\bar{\rho}} \mathcal{H} - \\ &\partial_{\bar{\nu}} \omega^{\bar{\mu}\alpha} \omega^{\bar{\nu}\rho} \partial_{\bar{\mu}} \mathcal{H} \partial_\rho \mathcal{H} - \omega^{\bar{\mu}\alpha} \omega^{\nu\bar{\rho}} \partial_{\bar{\rho}} \mathcal{H} \partial_{\bar{\mu}\nu} \mathcal{H} - \omega^{\bar{\mu}\alpha} \omega^{\bar{\nu}\rho} \partial_\rho \mathcal{H} \partial_{\bar{\mu}\bar{\nu}} \mathcal{H}. \\ &= a(t) \omega^{\bar{\mu}\alpha} \partial_{\bar{\mu}} \mathcal{H} + b(t) J_\mu^\alpha \omega^{\bar{\nu}\mu} \partial_{\bar{\nu}} \mathcal{H} = (a(t) + ib(t)) \omega^{\bar{\mu}\alpha} \partial_{\bar{\mu}} \mathcal{H}. \end{aligned} \quad (10)$$

Using the identities $\omega_{\alpha\bar{\beta}} = i g_{\alpha\bar{\beta}}$, $\Gamma_{\mu\nu}^\alpha = g^{\alpha\bar{\rho}} \partial_\mu g_{\nu\bar{\rho}} = -\omega^{\alpha\bar{\rho}} \partial_\mu \omega_{\nu\bar{\rho}}$ and Eqs. (1), (2) we can write (10) in the following form

$$\mathcal{H}_{,\alpha\bar{\mu}} g^{\rho\bar{\mu}} \mathcal{H}_{,\rho} + \mathcal{H}_{,\alpha\mu} g^{\mu\bar{\rho}} \mathcal{H}_{,\bar{\rho}} = (b - ia) \mathcal{H}_{,\alpha}. \quad (11)$$

where comma denotes covariant differentiation with respect to the metric g . Taking into account the reality of the function \mathcal{H} and Eqs. (1) – (3) we obtain (6) where functions A and B coincide with $a(t)$, $b(t)$ for any trajectory γ_t of H -planar flow γ .

Let we have a general solution of the equation

$$\nabla_\chi \chi = b(t) \chi - a(t) J \chi, \quad \chi = \frac{d\gamma}{dt}$$

of H -planar curve in a $2n$ -dimensional Kähler manifold M :

$$x^i = x^i(t, c_1^l, c_2^l), \quad i, l = 1, \dots, 2n \quad (12)$$

where c_1^l, c_2^l are integration constants.

Now we propose the method of finding general solution of Eq. (6).

If we take $2n + 1$ numbers of c_1^l, c_2^l , $l = 1, \dots, 2n$ to be constant and vary rest $2n - 1$ numbers we obtain a family of H -planar curves in an open domain $U \subset M$

$$x^i = x^i(t, \sigma^A), \quad A = 1, \dots, 2n - 1. \quad (13)$$

Let us calculate the time derivative of Hamiltonian \mathcal{H} along the curves of this family

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial x^i} \frac{dx^i}{dt} = g_{ik} \frac{dx^i}{dt} \frac{dx^k}{dt}$$

then

$$\mathcal{H} = \int dt g_{ik} \frac{dx^i}{dt} \frac{dx^k}{dt} + \text{const.}$$

By using (13) we can eliminate t and σ^A , $A = 1, \dots, 2n - 1$ from these expressions and obtain Hamiltonian \mathcal{H} as a function of coordinates x^i , $i = 1, \dots, 2n$. Considering different families of H -planar curves of the form (13) we receive different Hamiltonians obeying (6).

Since any Hamilton flow with a Hamiltonian \mathcal{H} defines a $2n - 1$ -parameter family of H -planar curves we described the way of construction general solution of Eq. (6) using general solution of equation of H -planar curve in a $2n$ -dimensional Kähler manifold M .

In the conclusion of this section let us consider gradient flow γ_{gr} of the function \mathcal{H} :

$$\frac{d\gamma}{dt} = \text{grad } \mathcal{H}. \quad (14)$$

Writing (11) for trajectories of the system (14) we find

$$\nabla_\chi \chi = b(t)\chi + a(t)J\chi, \quad \chi = \frac{d\gamma}{dt}.$$

From here it is seen that considering trajectories are H -planar curves. It yields the following

Theorem 3 *Let γ be H -planar Hamilton flow with Hamiltonian \mathcal{H} on Kähler manifold M . Then trajectories of gradient flow γ_{gr} of function \mathcal{H} are H -planar curves.*

3 Magnetic fields on Riemannian manifolds

Magnetic flows on (pseudo) Riemannian manifolds are interesting example of dynamical systems whose trajectories (trajectories of charged particles in magnetic field) in general case are not geodesics but closely related with Riemannian structure of the manifold. If considered manifold is Kähler one there is an interesting example of magnetic fields for which these trajectories are H -planar curves — Kähler magnetic fields.

Let (M, g) be m -dimensional pseudo Riemannian manifold. Closed 2-form B on M is said to be *magnetic field*. If M is Lorentzian manifold then magnetic fields on M usually refers to as *electromagnetic*.

Let \mathcal{I} be smooth field of skew-symmetric linear operators $\mathcal{I}_p : T_p M \rightarrow T_p M$ defined by the condition

$$B(X, Y) = g(\mathcal{I}X, Y) \quad (15)$$

for any vector fields X, Y on M . A smooth curve $\gamma : I = [0, 1] \rightarrow M$ is called *trajectory of magnetic field B* if its tangent vector $\chi \equiv d\gamma(t)/dt$ satisfies the equation

$$\nabla_\chi \chi = \mathcal{I}(\chi). \quad (16)$$

Since operator \mathcal{I} is skew-symmetric, any trajectory γ of magnetic field B on M has constant speed, i.e.

$$\frac{d}{dt}g(\chi, \chi) = 2g(\mathcal{I}(\chi), \chi) = 0. \quad (17)$$

If $\gamma(t)$ is trajectory of magnetic field B with the speed V then $\gamma(\alpha t)$, $\alpha \in \mathbf{R}$ is trajectory of magnetic field αB with the same speed.

The magnetic field B on pseudo Riemannian manifold (M, g) is called *uniform* [1, 2, 4] if tensor field \mathcal{I} is parallel with respect to Riemannian connection ∇ of the metric g : $\nabla \mathcal{I} = 0$. In the case of flat manifold M such definition coincides with the definition of a homogeneous magnetic field in the flat space.

Let (M, g, J) be a Kähler manifold of real dimension $2n$ and ω is fundamental 2-form of this manifold. Closed 2-form $q\omega$ where $q \neq 0$ is a real constant is uniform magnetic field which we call *Kähler magnetic field* on M [1, 2].

Trajectories of uniform magnetic fields on surfaces of constant curvature have studied Comtet [4] and Sunada [12]. They have proven that on the sphere S^2 and on Euclidean plane \mathbf{R}^2 these trajectories are circles (in the first case they are circles in the sense of Euclidean geometry of \mathbf{R}^3 in which S^2 is canonically imbedded). On the hyperbolic plane of sectional curvature $-k$ the behaviour of trajectories principally differs. If $|q| \geq \sqrt{k}$, then unit speed trajectories are still closed curve, but when $|q| \leq \sqrt{k}$ the trajectories of unit speed are open curve. Adachi [1, 2] has generalized these results on the case of properly Kähler manifolds of any dimensions.

The studies of Kähler magnetic fields on pseudo Riemannian manifolds is of interest from the point of view of applications in theoretical physics, because curves described by the equation (16) in four-dimensional manifold can be interpreted as trajectories of charged particles in electromagnetic and gravitational fields. Similar curves in pseudo Riemannian and, in particular, Kähler manifolds of any signature and dimension arise as trajectories of particles in various field theories [14, 5, 10].

In the next section we investigate trajectories of Kähler magnetic fields on a Kähler manifold of constant holomorphic sectional curvature and any signature and dimension (see also [6, 7]).

4 Kähler magnetic fields on manifolds of constant holomorphic sectional curvature

Let M be Kähler manifold of constant holomorphic sectional curvature k with Kähler metric g and fundamental 2-form ω and let $B = q\omega$, $q \in \mathbf{R}$ is Kähler magnetic field on M . The trajectory γ_t of magnetic field B satisfies the equation (16) which in local complex coordinates z^α, \bar{z}^α , $\alpha = 1, \dots, n$ on M takes the form

$$\nabla_t \frac{dz^\alpha}{dt} = iq \frac{dz^\alpha}{dt}, \quad \nabla_t \frac{d\bar{z}^\alpha}{dt} = -iq \frac{d\bar{z}^\alpha}{dt} \quad (18)$$

where ∇ is covariant derivative with respect to the metric g and $\nabla_t \equiv dz^\sigma/dt \nabla_\sigma + d\bar{z}^\sigma/dt \nabla_{\bar{\sigma}}$.

According to Theorem 1 it exists H -projective mapping $f : \mathbf{C}^n \rightarrow M$. Since H -projective mapping preserve the complex structure it is possible to choose on \mathbf{C}^n complex coordinates in which all Cristoffel symbols of the metric are equal to zero. Then in corresponding local complex coordinates on M Cristoffel symbols are defined by the following formula

$$\Gamma_{\alpha\mu}^\lambda = \psi_\alpha \delta_\mu^\lambda + \psi_\mu \delta_\alpha^\lambda \quad (19)$$

where $\psi_\alpha = \partial_\alpha \psi$ and ψ is a smooth function on M . Kähler manifolds of constant holomorphic sectional curvature are Einstein manifolds and, hence [8],

$$R_{\alpha\bar{\beta}} = \frac{1}{2}k(n+1)g_{\alpha\bar{\beta}}, \quad R_{\alpha\beta} = 0, \quad R_{\bar{\alpha}\bar{\beta}} = 0, \quad (20)$$

where R_{ij} are components of Ricci tensor. From the equalities $R_{\alpha\bar{\beta}} = \partial_{\bar{\beta}} \Gamma_{\alpha\sigma}^\sigma$, $\overline{R_{\alpha\bar{\beta}}} = R_{\bar{\alpha}\beta}$ (see, for example, [8]) using (19) and (20) we obtain

$$\partial_{\alpha\bar{\beta}} \psi = -\frac{k}{4} \partial_{\alpha\bar{\beta}} \Phi.$$

Hence,

$$\psi = -\frac{k}{4}\Phi$$

up to the function of the form $\varphi(z) + \overline{\varphi}(\overline{z})$ which is omitted because it corresponds to admissible transformations of Kähler of potential $\Phi \rightarrow \Phi + \frac{k}{4}(\varphi(z) + \overline{\varphi}(\overline{z}))$. Using Theorem 1 from here we find

$$\psi = -\frac{1}{2} \ln \left(\sum_{\alpha} \epsilon_{\alpha} z^{\alpha} \overline{z^{\alpha}} \right) + 1, \quad \epsilon_{\alpha} = \pm 1. \quad (21)$$

Let γ be H -planar curve determined by Eq. (18) and $f : M \rightarrow M'$ be H -projective mapping. In respective complex coordinates the curve $f^{-1}\gamma$ is defined by the same equation as γ (i.e. $z^{\alpha} = z^{\alpha}(t)$, $\overline{z^{\alpha}} = \overline{z^{\alpha}}(t)$, $\alpha = 1, \dots, n$) and satisfies the equation of H -planar curve in \mathbf{C}^n :

$$\frac{d^2 z^{\alpha}}{dt^2} = c(t) \frac{dz^{\alpha}}{dt}, \quad \frac{d^2 \overline{z^{\alpha}}}{dt^2} = \overline{c(t)} \frac{d\overline{z^{\alpha}}}{dt} \quad (22)$$

where $c(t)$ is a smooth complex-valued function.

Thus, the dynamical system (18) can be simulated dynamical system (22). Solving equation (22) we find.

$$z^{\alpha} = C_1^{\alpha} f(t) + C_2^{\alpha} \quad (23)$$

where $f(t) = \int dt (\exp \int c(t) dt)$, and C_1^{α} , C_2^{α} are integration constants.

Let us consider two cases: $\sum_{\alpha} \epsilon_{\alpha} C_1^{\alpha} \overline{C_1^{\alpha}} \neq 0$ and $\sum_{\alpha} \epsilon_{\alpha} C_1^{\alpha} \overline{C_1^{\alpha}} = 0$. In the first case making in (23) the following transformation

$$\tilde{f}(t) = v f(t) + w, \quad v, w \in \mathbf{C} \quad (24)$$

and presenting z^{α} in the form $z^{\alpha} = \tilde{C}_1^{\alpha} \tilde{f}(t) + \tilde{C}_2^{\alpha}$, it is possible to choose constants v , w in such a way that

$$\sum_{\alpha} \epsilon_{\alpha} \tilde{C}_1^{\alpha} \overline{\tilde{C}_2^{\alpha}} = 0, \quad \sum_{\alpha} \epsilon_{\alpha} \tilde{C}_1^{\alpha} \overline{\tilde{C}_1^{\alpha}} = \mathcal{A}, \quad (25)$$

where $\mathcal{A} = \text{sgn} \left(\sum_{\alpha} \epsilon_{\alpha} C_1^{\alpha} \overline{C_1^{\alpha}} \right)$.

By the same way in the case $\sum_{\alpha} \epsilon_{\alpha} C_1^{\alpha} \overline{C_1^{\alpha}} = 0$ making transformation (24) one can choose constant v and w so that

$$\sum_{\alpha} \epsilon_{\alpha} \tilde{C}_2^{\alpha} \overline{\tilde{C}_2^{\alpha}} = -1, \quad \sum_{\alpha} \epsilon_{\alpha} \tilde{C}_1^{\alpha} \overline{\tilde{C}_2^{\alpha}} = 1. \quad (26)$$

Hereinafter we shall consider that conditions (25) or (26) holds and tilde over C_1^{α} , C_2^{α} and f will be omitted.

Let us present complex valued function $f(t)$ in the following form

$$f(t) = \exp(r(t) + i\phi(t)), \quad (27)$$

where $r(t) = \overline{r(t)}$ and $\phi(t) = \overline{\phi(t)}$. From (24) it follows that trajectories γ_t of Kähler magnetic field $B = q\omega$ belong to 2-dimensional submanifold $P \subset M$ defined in the complex coordinates z^α , $\alpha = 1, \dots, n$ by the equations

$$z^\alpha = C_1^\alpha u + C_2^\alpha, \quad u \in \mathbf{C}. \quad (28)$$

Thus, the functions r and ϕ can be treated as coordinates of the trajectory's point on the surface P . Since any geodesics is a trajectory of Kähler magnetic field $B_0 = q_0\omega$, $q = 0$ it is easy to see that P is properly geodesic submanifold in M . Using the formula [8]

$$\Gamma_{\beta\gamma}^\alpha = \overline{\Gamma_{\beta\gamma}^\alpha} = g^{\alpha\bar{\mu}} \partial_\beta g_{\bar{\mu}\gamma},$$

from (18) we receive

$$\ddot{z}^\alpha = -4\partial_\nu \psi \dot{z}^\nu \dot{z}^\alpha + iq\dot{z}^\alpha \quad (29)$$

where dot denotes derivation on parameter t .

So we obtain n complex equations which can be rewritten as $2n$ real ones

$$\begin{aligned} \frac{d}{dt} \ln(\dot{z}^\alpha \dot{\bar{z}}^\alpha) &= -4 \frac{d\psi}{dt}, \\ i \frac{d}{dt} \ln \frac{\dot{z}^\alpha}{\dot{\bar{z}}^\alpha} &= -4i(\partial_\nu \psi \dot{z}^\nu - \partial_{\bar{\nu}} \psi \dot{\bar{z}}^{\bar{\nu}}) - 2q, \end{aligned} \quad (30)$$

where the summation on $\underline{\alpha}$ is not carried out.

In the case $\sum_\alpha \epsilon_\alpha \tilde{C}_1^\alpha \tilde{C}_2^\alpha = 0$, $\sum_\alpha \epsilon_\alpha \tilde{C}_1^\alpha \tilde{C}_1^\alpha = \mathcal{A}$ from here with the help of (21), (23) and (27) we receive

$$\frac{d}{dt} \ln(e^{2r}(\dot{r}^2 + \dot{\phi}^2)) = -4 \frac{d\psi}{dt}, \quad (31)$$

$$i \frac{d}{dt} \ln \frac{\dot{r} + i\dot{\phi}}{\dot{r} - i\dot{\phi}} - 2\dot{\phi} = -4i(\partial_\nu \psi \dot{z}^\nu - \partial_{\bar{\nu}} \psi \dot{\bar{z}}^{\bar{\nu}}) - 2q, \quad (32)$$

Using Eqs. (21), (23), (25) and (27) from (31), (32) we find

$$\frac{d}{dt} \ln(e^{2r}(\dot{r}^2 + \dot{\phi}^2)) = 2 \frac{d}{dt} \ln(\mathcal{A}e^{2r} + C + 1), \quad (33)$$

$$i \frac{d}{dt} \ln \frac{\dot{r} + i\dot{\phi}}{\dot{r} - i\dot{\phi}} - 2\dot{\phi} = -4(\mathcal{A}e^{2r} + C + 1)^{-1} \mathcal{A}e^{2r} \dot{\phi} - 2q, \quad (34)$$

where $C = \sum_{\alpha} \epsilon_{\alpha} C_2^{\alpha} \overline{C_2^{\alpha}}$. Integrating (33) we receive

$$\mathcal{J}^2 e^{2r} (\dot{r}^2 + \dot{\phi}^2) = (\mathcal{A}e^{2r} + C + 1)^2 \quad (35)$$

where \mathcal{J} is an integration constant.

Using (22) it is possible to calculate a speed square $V = 2g_{\alpha\bar{\beta}} \dot{z}^{\alpha} \dot{z}^{\bar{\beta}}$ of trajectory $z^{\alpha}(t)$

$$V = \frac{4}{k} \frac{(\dot{r}^2 + \dot{\phi}^2) e^{2r} (C + 1)}{(\mathcal{A}e^{2r} + C + 1)^2}.$$

From here with the help of (35) it follows

$$V = \frac{4}{k} \frac{C + 1}{\mathcal{J}^2},$$

where k is holomorphic sectional curvature of manifold M . According to (17) any trajectory γ of magnetic field B has constant speed, hence, $V = \text{const.}$ So we find that *for a trajectory of Kähler magnetic field with fixed value of C the integration constant \mathcal{J} is inverse proportional to square root of speed.*

Let us consider the coordinate r as function of coordinate ϕ on the surface $P \in M : r = r(\phi)$. We choose ϕ to be an independent variable and denote

$$p \equiv \frac{d\phi}{dt}, \quad r' \equiv \frac{dr}{d\phi} \quad \text{then} \quad \frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = r' p.$$

Then Eqs. (33) and (34) takes the following form

$$2p \left(\frac{r''}{r'^2 + 1} - 1 \right) = -4(\mathcal{A}e^{2r} + C + 1)^{-1} \mathcal{A}e^{2r} p - 2q, \quad (36)$$

$$e^{2r} p^2 (r'^2 + 1) \mathcal{J}^2 = (\mathcal{A}e^{2r} + C + 1)^2.$$

Expressing p from the last equation and substituting it in (36) we receive

$$\begin{aligned} \frac{r''}{r'^2 + 1} &= 1 - (\mathcal{A}e^{2r} + C + 1)^{-1} 2\mathcal{A}e^{2r} \\ &\quad - q(\mathcal{A}e^{2r} + C + 1)^{-1} \mathcal{J} e^r (r'^2 + 1)^{1/2}. \end{aligned} \quad (37)$$

Similarly, in the case when $\sum_{\alpha} \epsilon_{\alpha} C_1^{\alpha} \overline{C_1^{\alpha}} = 0$ from (21), (23), (27) (31) and (32) we find

$$\frac{r''}{r'^2 + 1} = -r' \text{tg } \phi - q \mathcal{J} (1 + r'^2)^{1/2} (\cos \phi)^{-1}. \quad (38)$$

Theorem 4 *Let (M, g, J) be geodesically complete Kähler manifold of constant holomorphic sectional curvature k with fundamental 2-form ω and $B = q\omega$, $q \in \mathbf{R}$ be Kähler magnetic field on M . Let z^α, \bar{z}^α , $\alpha = 1, \dots, n$ be such local complex coordinates on M that Cristoffel symbols of metric g have the form (19). Then any trajectory $\gamma(t)$ of magnetic field B belongs to 2-dimensional properly geodesic submanifold $P \subset M$ determined in coordinates $(z^\alpha, \bar{z}^\alpha)$ by the equations*

$$z^\alpha = C_1^\alpha u + C_2^\alpha, \quad u \in \mathbf{C}.$$

The coordinates r, ϕ of the point $p = \gamma_t \in P \subset M$ satisfy the equation (37) if $\mathcal{A} = \sum_\sigma C_1^\sigma \bar{C}_1^\sigma \neq 0$ and equation (38) if $\sum_\sigma C_1^\sigma \bar{C}_1^\sigma = 0$.

This theorem reduces the study of of equations (18) to the consideration of one ordinary differential equation of the second order. The solution of this equation can be obtained by numerical methods.

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